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Super W_n Gravity on Compactified Moduli Space

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We study the divergent behavior of W gravity theories. As a tool, we use the Grothendieck–Riemann–Roch theorem on the compactified moduli space. We show that W_n gravity has severe divergences caused by negative masses. However, for superextension of W_n gravity the divergences by negative masses are miraculously cured by the counterpart contribution of superpartners.

1. INTRODUCTION

The Virasoro algebra has played an important role in the understanding of string theory and conformal field theory. On the other hand, over the years there have been many attempts to generalize theories of classical and quantum gravity to higher spin extensions which include fields of spin greater than 2. The high-spin extension of Virasoro algebra is known as W-algebra (Zamolodchikov, 1985; Fateev and Lukyanov, 1988). When W-algebra serves as a gauge algebra to two-dimensional gravity, W gravity theory is obtained (Matsuo, 1989; Hull, 1990; Pawelczyk, 1991). A well-known example based on Zamolodchikov's W_3 algebra is W_3 gravity. Since W-algebra seems to be a fundamental symmetry in two-dimensional gravity, the study of W-gravity theories is very important.

For a theory to be a consistent quantum field theory, it should satisfy certain criteria; for instance, the absence of anomaly and the disappearance of divergences caused by negative masses.

Let us take the example of bosonic string theory, which contains a spectrum of various states, including tachyons. If we take a long, thin tube of constant diameter and total length L, we can canonically quantize it on slices of constant time. Then the loop amplitude will be the trace over Fock

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space of the amplitude $e^{-E_n L}$, where E_n is the energy of states n. For the lowest energy $E_0 \leq 0$, the integral $\int dL\, e^{-E_0 L}$ will diverge.

A few approaches are known for studying the divergence behavior of 2D theory over a long, thin tube. Among them a powerful tool is the Grothendieck–Riemann–Roch theorem on the compactified moduli space (Harshorne, 1977; Deligne and Mumford, 1969; Mumford, 1977; Harris and Mumford, 1982). Since it was fully explained in Kwon (1991), here we only briefly sketch the outline.

First, an important point is that the appropriate g-loop thin-tube geometry can be identified with the compactified moduli space $\tilde{S}_g$ to g-genus. The compactified moduli space $\tilde{S}_g$ is the moduli space of stable curves. The boundary $[\Delta] = \tilde{S} - S$ is the union of components $[\Delta] = \Delta_0, \ldots, \Delta_{g/2}$ containing surfaces with nodes of degree $0 \leq i \leq [g/2]$.

Second, the negative mass pole in the vacuum-to-vacuum amplitude has a direct relationship with a nonvanishing Chern class on $\tilde{S}_g$. The nonvanishing Chern class means the existence of a pole on the boundary of $\tilde{S}$, which implies the existence of a negative mass pole (Kwon, 1990, 1991). To evaluate the Chern class on $\tilde{S}$, we can use the Grothendieck–Riemann–Roch (GRR) theorem on the compactified moduli space. The theorem can be given as follows:

$$C_1(d(W_f^{\otimes n})) = (6n^2 - 6n + 1)C_1(\det f_* W_f) - [n(n-1)/2]\delta \tag{1}$$

Here $d(L) = \det(f_* L) \otimes \det(f_*(L^{-1} \otimes W_f))$, f is the map from the universal curve of moduli space to moduli space, W_f is the relative canonical bundle of f on the compactified moduli space, and δ is the compactified divisor class.

In this paper our strategy to study the divergent behavior of W gravity is to evaluate the Chern class on $\tilde{S}_g$, using GRR theorem.

2. *W* GRAVITY

First let us consider W_3 gravity. The action of W_3 gravity for a spin-2 gauge field h_{--} and a spin 3-gauge field B_{---} in two-dimensional Euclidean space with coordinates $X^{\pm} = \sigma \pm i\tau$ coupled to $i = 1, \ldots, n$ real scalar fields ϕ^i is given by (Matsuo, 1989; Hull, 1990; Pawelczyk, 1991; Schoutens *et al.*, 1991a,b; Ceresole *et al.*, 1968)

$$I = \int d^2x \left[\frac{1}{2}\,\partial_+\phi^i\partial_-\phi^i - \frac{1}{2}\,h_{--}\partial_+\phi^i\partial_+\phi^i \right.$$
$$\left. -\frac{1}{3}\,B_{---}\partial_+\phi^i\partial_+\phi^i\partial_+\phi^k d^{ijk}\right] \tag{2}$$

If d^{ijk} satisfies the quadratic relation

$$d^{i(jk}d^{l)mi} = \delta^{(jk}\delta^{l)m} \tag{3}$$

the action is invariant under the following local $\epsilon_-(x^+, x^-)$ transformations and local $\lambda_{--}(x^+, x^-)$ transformations:

$$\begin{aligned}
\delta\phi^i &= \epsilon_-\partial_+\phi^i + \lambda_{--}\partial_+\phi^j\partial_+\phi^k d^{ijk} \\
\delta h_{--} &= \partial_-\epsilon_- - h_{--}\partial_+\epsilon_- + \epsilon_-\partial_+h_{++} \\
&\quad + (\lambda_{--}\partial_+B_{--} - B_{---}\partial_+\partial_+\lambda_{--})\partial_+\phi^i\partial_+\phi^i \\
\delta B_{---} &= \epsilon_-\partial_+B_{---} - 2B_{---}\partial_+\epsilon_- \\
&\quad + \partial_-\lambda_{--} - h_{--}\partial_+\lambda_{--} + 2\lambda_{--}\partial_+h_{--}
\end{aligned} \tag{4}$$

Now let us quantize W_3 gravity. For gauge fixing, the symmetries can be used to choose the conformal gauge $h_{--} = 0$, $B_{---} = 0$ locally. Since the action is invariant only on shell, the quantization procedure of Batalin and Vilkovisky can be used.

The ghost action for W_3 gravity is given by

$$S^{\mathrm{gh}} = \frac{1}{2}\int d^2x\,(b_{++}\partial_-c_- + u_{+++}\partial_-v_{--}) \tag{5}$$

where b_{++}, c_- are the usual conformal antighost and ghost, while u_{+++}, v_{--} are the antighost and ghost for the λ symmetry.

By (2) and (5), the partition function for W_3 gravity can be obtained, and is proportional to[3]

$$Z \propto |\det \nabla_0|^{-m}|\det \nabla_2|^2|\det \nabla_3|^2 \tag{6}$$

This is the square modulus of a section of a line bundle

$$B = L^{-m/2} \times P_2 \times P_3 \tag{7}$$

where L, P_2, and P_3 are the line bundles associated with the operators ∇_0, ∇_2, and ∇_3. The Chern class for each line bundle can be evaluated by the GRR theorem;

$$\begin{aligned}
C_1(P_2) &= 13C_1(L) - \delta \\
C_1(P_3) &= 37\underset{\text{(i)}}{C_1(L)} - \underset{\text{(ii)}}{3\delta}
\end{aligned} \tag{8}$$

The absence of anomaly requires the value of m to be $m = -100$, which can be determined by (8)-(i). However for $m = -100$, the line bundle (7)

[3]The full geometrical structure of the measure to W gravity theories is not known and only the measure by ghost and matter sectors will be considered.

develops a nonvanishing Chern class -4δ. As explained, the nonvanishing Chern class for the line bundle (7) implies that there is a pole on the boundary of $\tilde{S}$. So we can expect some badly behaved divergence by negative masses in W_3 gravity. Now it is quite natural to ask whether the situation of the divergent behavior is improved in W_n gravity. The answer to this question is "no." To see this, let us consider W_n gravity. The line bundle whose square modulus of a section is the partition function of W_n gravity is given, using the Batalin and Vilkovisky method, by

$$B = L^{-m/2} \times \cdots \times P_n \tag{9}$$

By the requirement of the absence of anomaly, the critical value of m is given by the first part of the GRR theorem. But it can be shown that for the value of m, the degree of nonvanishing Chern class, which can be obtained by the second part of the GRR theorem, grows more rapidly as n gets larger.

Physically this can be understood in the following way. As the large-spin particles are included in the theory, the energy spectrum contains some large negative value. Therefore, as we have seen the divergent behavior of the bosonic string on the long, thin tube, the large negative masses cause a severe divergence. This is what we saw using the GRR theorem.

It can be expected that since supersymmetry does not allow the energy spectrum to have a negative mass, the supertheory should not develop any pole on $\tilde{S}$.

To see whether our conjecture is true or not, let us consider the superextension of W_n gravity. In the case of super W_n gravity the line bundle can be given in the following way (we consider only $N = 1$ supersymmetry):

$$\begin{aligned} B = {} & L^{-m/2} \times (E_\alpha)^{m/2} \times ((S_{3/2})_\beta)^{-1} \times P_2 \times ((S_{5/2})_\gamma)^{-1} \\ & \times P_3 \times ((S_{7/2})_\delta)^{-1} \times P_4 \times \cdot \times ((S_{(2n-1)/2})_\sigma^{-1} \times P_n \end{aligned} \tag{10}$$

Here E_α, $(S_{3/2})_\alpha$, $(S_{5/2})_\alpha$, etc., are the line bundles associated with operators $\nabla_{1/2}$, $\nabla_{3/2}$, $\nabla_{5/2}$, etc., whose Chern class is given by

$$\begin{aligned} C_1(E_\alpha) &= -\frac{1}{2} C_1(L) + \frac{1}{8}\delta \\ C_1((S_{3/2})\alpha) &= \frac{11}{2} C_1(L) - \frac{3}{8}\delta \\ C_1((S_{5/2})_\alpha) &= \frac{47}{2} C_1(L) - \frac{15}{8}\delta \\ &\vdots \end{aligned} \tag{11}$$

The critical value of m is given by

$$-\frac{3}{4}m + \sum_{k=2}^{n}\left(6k - \frac{9}{2}\right) = 0 \tag{12}$$

As expected, the above value of m does not develop any nonvanishing Chern class δ, which proves our conjecture.

Physically this means that as the large-spin particles are included in the theory, the energy spectrum contains some large negative value; however, the divergences by negative masses are miraculously cured by the counterpart contribution of superpartners.

3. CONCLUSION

It has been shown that super-W_n gravity would not develop any nonvanishing Chern class, which would imply no tachyon poles. The GRR theorem can be a powerful tool to distinguish the divergent behaviors of a theory without calculating loop diagrams in 2 dimensions. Using the GRR theorem effectively, one can construct new theories without divergence terms.

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